Chapter 1

THE PASCAL TRIANGLE

In modern mathematics, more and more stress is placed on the context in which statements are true. In elementary mathematics this generally means an emphasis on a clear understanding of which number systems possess certain properties. We begin, then, by describing briefly the number systems with which we will be concerned. These number systems have developed through successive enlargements of previous systems.

At one time a "number" meant one of the **natural numbers**: 1,2,3,4,5,6,... . The next numbers to be introduced were the fractions:1/2, 1/3, 2/3, 1/4, 3/4, 1/5,..., and later the set of numbers was expanded to include zero and the negative integers and fractions. The number system consisting of zero and the positive and negative integers and fractions is called the system of **rational numbers**, the word "rational" being used to indicate that the numbers are ratios of integers. The integers themselves can be thought of as ratios of integers since 1 = 1/1, -1 = -1/1, 2 = 2/1, -2 = -2/1, 3 = 3/1, etc.

The need to enlarge the rational number system became evident when mathematicians proved that certain constructible lengths, such as the length $\sqrt{2}$ of a diagonal of a unit square, *cannot* be expressed as rational numbers. The system of **real numbers** then came into use. The real numbers include all the natural numbers; all the fractions; numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{(2+\sqrt{6})/3}$ which represent constructible lengths; numbers such as $\sqrt[3]{2}$ and π which do not represent lengths constructible from a given unit with compass and straightedge; and the negatives of all these numbers. Modern technology and science make great use of a still larger number system, called the **complex numbers**, consisting of the numbers of the form $a + bi$ with a and b real numbers and $i^2 = -1$.

Our first topic is the Pascal Triangle, an infinite array of natural numbers. We begin by considering expansions of the powers $(a + b)^n$ of a sum of two terms. Clearly, $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$. Then $(a + b)^3 = (a + b)^2(a + b) = (a^2 + 2ab + b^2)(a + b)$. We expand this last expression as the sum of all products of a term of $a^2 + 2ab + b^2$ by a term of $a + b$ in the following manner:

$$\begin{array}{llll}
a^2 & + 2ab & + b^2 & \\
a & + b & & \\
\hline
a^3 & + 2a^2b & + ab^2 & \\
 & a^2b & + 2ab^2 & + b^3 \\
\hline
a^3 & + 3a^2b & + 3ab^2 & + b^3
\end{array}$$

Hence $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. If $a \neq 0$ and $b \neq 0$ (a is not equal to zero and b is not equal to zero), this may be written

$$(a + b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

The terms of the expanded form are such that the exponent for a starts as 3 and decreases by one each time, while the exponent of b starts as 0 and increases by one each time. Thus the sum of the exponents is 3 in each term.

One might guess that by analogy the expansion of $(a + b)^4$ involves a^4, a^3b, a^2b^2, ab^3, and b^4. This is verified by expanding $(a + b)^4 = (a + b)^3(a + b) = (a^3 + 3a^2b + 3ab^2 + b^3)(a + b)$ as follows:

(1)	a^3	$+ 3a^2b$	$+ 3ab^2$	$+ b^3$	
(2)	a	$+ b$			
(3)	a^4	$+ 3a^3b$	$+ 3a^2b^2$	$+ ab^3$	
(4)		a^3b	$+ 3a^2b^2$	$+ 3ab^3$	$+ b^4$
(5)	a^4	$+ 4a^3b$	$+ 6a^2b^2$	$+ 4ab^3$	$+ b^4$

Thus we see that a^4, a^3b, a^2b^2, ab^3, and b^4 are multiplied by 1, 4, 6, 4, 1 to form the terms of the expansion. The numbers 1, 4, 6, 4, 1 are the coefficients of the expansion. Examination of expressions (1) to (5), above, shows that these coefficients are obtainable from the coefficients 1, 3, 3, 1 of $(a + b)^3$ by means of the following condensed versions of (3), (4), and (5):

(3*)	1	3	3	1	
(4*)		1	3	3	1
(5*)	1	4	6	4	1

We now tabulate the coefficients of $(a + b)^n$ for $n = 0, 1, 2, 3, 4, \ldots$ in a triangular array:

n	Coefficients of $(a + b)^n$
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
...	. . . . . .

One may observe that the array is bordered with 1's and that each number inside the border is the sum of the two closest numbers on the previous line. This observaion simplifies the generation of additional lines of the array. For example, the coefficients for $n = 5$ are $1, 1 + 4 = 5, 4 + 6 = 10, 6 + 4 = 10, 4 + 1 = 5$, and 1.

The above triangular array is called the **<u>Pascal Triangle</u>** in honor of the mathematician Blaise Pascal (1623-1662). A notation for the coefficients of $(a + b)^n$ is

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}. \tag{6}$$

For example, one writes $(a + b)^4$ in this notation as

$$\binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

where $\binom{4}{0} = 1 = \binom{4}{4}$, $\binom{4}{1} = 4 = \binom{4}{3}$, and $\binom{4}{2} = 6$.

A two-term expression is called a **<u>binomial</u>**, and an expansion for an expression such as $(a + b)^n$ is called a **<u>binomial</u>** **<u>expansion</u>**. The coefficients listed in (6) above are called **<u>binomial coefficients</u>**.

Note that the symbol $\binom{n}{k}$ denotes the coefficient of $a^{n-k}b^k$, or of a^kb^{n-k}, in the expansion of $(a + b)^n$. Thus $\binom{3}{1}$ is the coefficient 3 of a^2b or of ab^2 in the expansion of $(a + b)^3$,and $\binom{4}{2}$ is the coefficient 6 of x^2y^2 in $(x + y)^4$. One reads $\binom{n}{k}$ as "binomial coefficient n choose k" or simply as "n choose <u>k</u>." The reason for this terminology is given in Chapter 7.

In Figure 1, (on page 4) we see how n and k give us the location of $\binom{n}{k}$ in the Pascal Triangle. The number n in $\binom{n}{k}$ is the row number and k is the diagonal number if one adopts the convention of labeling the rows or diagonals as 0, 1, 2, ...

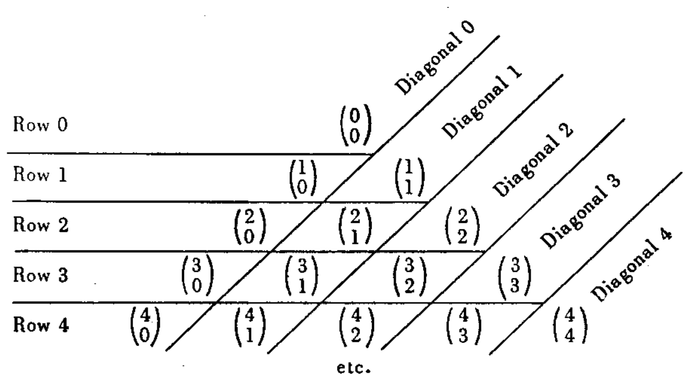

FIGURE 1

The formulas $\binom{n}{0} = 1 = \binom{n}{n}$ recall the fact that the Pascal Triangle is bordered with 1's. The rule that each number inside the border of 1's in the Pascal Triangle is the sum of the two closest entries on the previous line may be written as

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Example 1. Expand $(2x + 3y^2)^3$.

Solution: The expansion $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is an identity which remains true when one substitutes $a = 2x$ and $b = 3y^2$ and thus obtains

$$\begin{aligned}(2x + 3y^2)^3 &= (2x)^3 + 3(2x)^2(3y^2) + 3(2x)(3y^2)^2 + (3y^2)^3 \\ &= 8x^3 + 3(4x^2)(3y^2) + 3(2x)(9y^4) + 27\,y^6 \\ &= 8x^3 + 36x^2y^2 + 54xy^4 + 27y^6.\end{aligned}$$

Example 2. Show that $\binom{4}{0} + \binom{5}{1} + \binom{6}{2} + \binom{7}{3} + \binom{8}{4} = \binom{9}{4}$.

Solution: Using the fact that $\binom{4}{0} = 1 = \binom{5}{0}$ and the formula

$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ we see that

$$\binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{5}{1}+\binom{4}{0} = \binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{5}{1}+\binom{5}{0}$$

$$= \binom{8}{4}+\binom{7}{3}+\binom{6}{2}+\binom{6}{1}$$

$$= \binom{8}{4}+\binom{7}{3}+\binom{7}{2}$$

$$= \binom{8}{4}+\binom{8}{3}$$

$$= \binom{9}{4}.$$

Problems for Chapter 1

1. Give the value of $\binom{5}{2}$, that is, of the coefficient of a^3b^2 in $(a + b)^5$.

2. Give the value of $\binom{5}{4}$.

3. Find s if $\binom{5}{4} = \binom{5}{s}$ and s is not 4.

4. Find t if $\binom{5}{t} = \binom{5}{0}$ and t is not 0.

5. Obtain the binomial coefficients for $(a + b)^3$ from those for $(a + b)^2$ in the style of lines (3*), (4*), (5*) on page 2.

6. Obtain the binomial coefficients for $(a + b)^6$ from those for $(a + b)^5$ in the style of lines (3*), (4*), (5*) on page 2.

7. Generate the lines of the Pascal Triangle for $n = 6$ and $n = 7$, using the technique described at the top of page 3.

8. Find $\binom{8}{0}$, $\binom{8}{1}$, $\binom{8}{2}$, $\binom{8}{3}$, and $\binom{8}{4}$.

9. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{9}{7}$ and $\binom{9}{8}$.

10. Use $\binom{9}{1} = 9$ and $\binom{9}{2} = 36$ to find $\binom{10}{2}$ and $\binom{10}{8}$.

11. Expand $(5x + 2y)^3$.

12. Expand $(x^2 - 4y^2)^3$ by letting $a = x^2$ and $b = -4y^2$ in the expansion of $(a + b)^3$.

13. Show that:

(a) $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

(b) $(x - y)^5 = \binom{5}{0}x^5 - \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 - \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 - \binom{5}{5}y^5$.

14. Show that:

(a) $(x + 1)^6 + (x - 1)^6 = 2\left[\binom{6}{0}x^6 + \binom{6}{2}x^4 + \binom{6}{4}x^2 + \binom{6}{6}\right]$.

(b) $(x + y)^6 - (x - y)^6 = 2\left[\binom{6}{1}x^5y + \binom{6}{3}x^3y^3 + \binom{6}{5}xy^5\right]$.

15. Show that $(x + h)^3 - x^3 = h(3x^2 + 3xh + h^2)$.

16. Show that $(x + h)^{100} - x^{100} = h\left[\binom{100}{1}x^{99} + \binom{100}{2}x^{98}h + \binom{100}{3}x^{97}h^2 + \ldots + \binom{100}{100}h^{99}\right]$.

17. Find numerical values of c and m such that cx^3y^m is a term of the expansion of $(x + y)^8$.

18. Find d and n such that dx^5y^4 is a term of $(x + y)^n$.

19. Find each of the following:

(a) $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$.

(c) $2\binom{4}{0} + 2\binom{4}{1} + \binom{4}{2}$.

(d) $2\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2}\right]$.

(e) $2\left[\binom{6}{0} + \binom{6}{1} + \binom{6}{2}\right] + \binom{6}{3}$.

(f) $2\left[\binom{7}{7} + \binom{7}{6} + \binom{7}{5} + \binom{7}{4}\right]$.

20. Find the sum of the 101 binomial coefficients for $n = 100$ by assigning specific values to a and b in the identity

$$(a+b)^{100} = \binom{100}{0}a^{100} + \binom{100}{1}a^{99}b + \binom{100}{2}a^{98}b^2 + \ldots$$

$$+ \binom{100}{98}a^2b^{98} + \binom{100}{99}ab^{99} + \binom{100}{100}b^{100}.$$

21. Find each of the following:

(a) $\binom{2}{0} - \binom{2}{1} + \binom{2}{2}$.

(b) $\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}$.

22. Find each of the following:

(a) $\binom{100}{0} - \binom{100}{1} + \binom{100}{2} - \binom{100}{3} + \dots - \binom{100}{99} + \binom{100}{100}$.

(b) $\binom{101}{0} - \binom{101}{1} + \binom{101}{2} - \binom{101}{3} + \dots + \binom{101}{100} - \binom{101}{101}$.

23. Find each of the following:

(a) $\binom{4}{0} + \binom{4}{2} + \binom{4}{4}$.

(b) $\binom{5}{0} + \binom{5}{2} + \binom{5}{4}$.

(c) $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6}$.

(d) $\binom{7}{1} + \binom{7}{3} + \binom{7}{5} + \binom{7}{7}$.

24. Find each of the following:

(a) $\binom{1000}{0} + \binom{1000}{2} + \binom{1000}{4} + \dots + \binom{1000}{1000}$.

(b) $\binom{1000}{1} + \binom{1000}{3} + \binom{1000}{5} + \dots + \binom{1000}{999}$.

25. Find r, s, t, and u, given the following:

(a) $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{r}{3}$.

(b) $\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} = \binom{s}{4}$.

(c) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1} + \binom{5}{1} = \binom{t}{2}$.

(d) $\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} = \binom{u}{3}$.

26. Express $\binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \binom{7}{4} + \cdots + \binom{100}{4}$ as a binomial coefficient.

27. Express $\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \cdots + \binom{995}{990}$ as a binomial coefficient.

28. Show that

(a) $n = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \binom{3}{0} + \cdots + \binom{n-1}{0} = \binom{n}{1}$.

(b) $\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$.

(c) $\binom{n}{k} + 2\binom{n}{k+1} + \binom{n}{k+2} = \binom{n+2}{k+2}$ for $0 \le k \le n-2$.

29. Show that $\binom{9}{4}\binom{5}{3} = \binom{9}{3}\binom{6}{4} = \binom{9}{2}\binom{7}{3}$.

30. Show that $\binom{10}{1}\binom{9}{2}\binom{7}{3} = \binom{10}{4}\binom{6}{3}\binom{3}{2} = \binom{10}{2}\binom{8}{4}\binom{4}{1}$.

31. Expand $(x + y + z)^4$ by expanding $(w + z)^4$, then replacing w by $x + y$, and expanding further.

32. Expand $(x + y - z)^4$.

33. The sum of squares $\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$ is expressible in the form $\binom{2m}{m}$.

Find m.

34. Express each of the following in the form $\binom{2m}{m}$:

(a) $\binom{4}{0}\binom{4}{4} + \binom{4}{1}\binom{4}{3} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{1} + \binom{4}{4}\binom{4}{0}$.

(b) $2\left[\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2\right]$.

(c) $\binom{6}{3}^2 + 2\left[\binom{6}{2}^2 + \binom{6}{1}^2 + \binom{6}{0}^2\right]$.

35. Use $(x^3 + 3x^2 + 3x + 1)^2 = [(x + 1)^3]^2 = (x + 1)^6$ to show the following:

(a) $\binom{3}{0}^2 = \binom{6}{0}$.

(b) $\binom{3}{0}\binom{3}{1} + \binom{3}{1}\binom{3}{0} = \binom{6}{1}$.

(c) $\binom{3}{0}\binom{3}{2} + \binom{3}{1}\binom{3}{1} + \binom{3}{2}\binom{3}{0} = \binom{6}{2}$.

(d) $\binom{3}{0}\binom{3}{3} + \binom{3}{1}\binom{3}{2} + \binom{3}{2}\binom{3}{1} + \binom{3}{3}\binom{3}{0} = \binom{6}{3}$.

36. Express $\binom{100}{0}^2 + \binom{100}{1}^2 + \binom{100}{2}^2 + \cdots + \binom{100}{100}^2$ in the form $\binom{2m}{m}$.

37. How many of the 3 binomial coefficients $\binom{n}{k}$ with n = 0 or 1 are odd?

38. How many of the 10 binomial coefficients $\binom{n}{k}$ with n = 0, 1, 2, or 3 are odd?

39. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2, 3, 4, 5, 6, or 7, and how many of these are odd?

40. Show that all eight of the coefficients $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \ldots, \binom{7}{7}$ are odd.

41. For what values of *n* among 0, 1, 2,..., 20 are all the coefficients $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ on row *n* of the Pascal Triangle odd?

42. If *n* is an answer to the previous problem, how many of the binomial coefficients on row *n* + 1 of the Pascal Triangle are odd?

*43. How many binomial coefficients $\binom{n}{k}$ are there with n = 0, 1, 2,..., 1022, or 1023, and how many of these are even?